The Z-Transform

Z-Transform

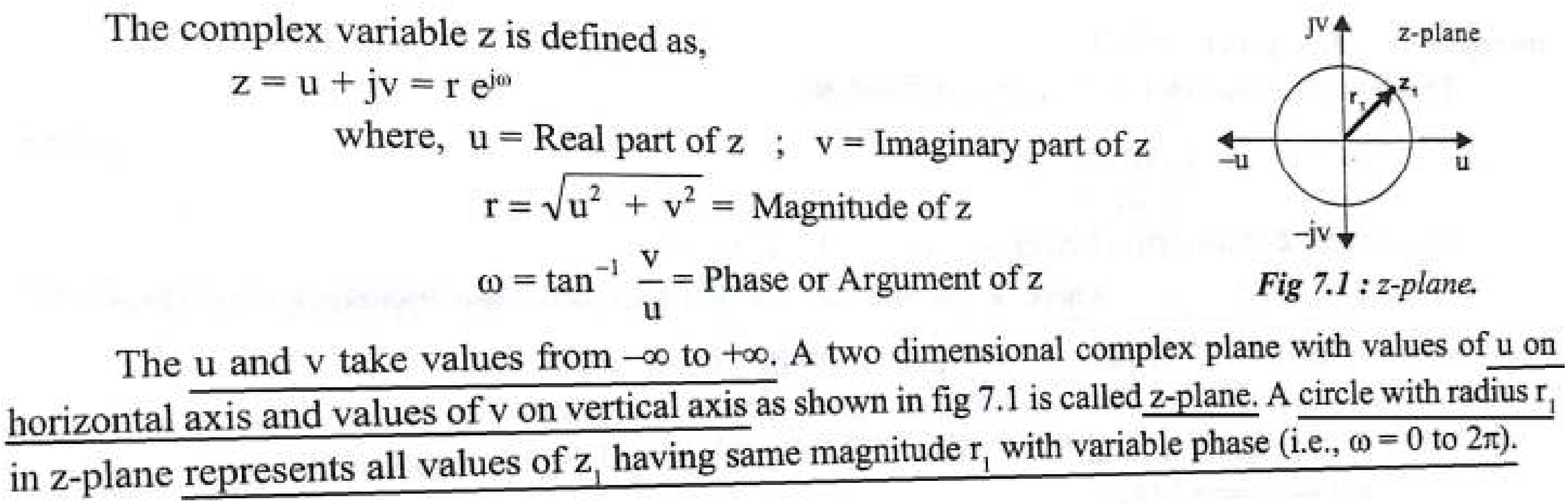
- Z-transform operator: $Z\{\cdot\}$

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

- The z-transform operator is seen to transform the sequence $x[n]$ into the function $X\{z\}$, where z is a continuous complex variable.
  - From time domain (or space domain, n-domain) to the z-domain

$$x[n] = \sum_{n=-\infty}^{\infty} x[t]\delta[t-n] \overset{z}{\leftrightarrow} Z\{x[n]\} = X(Z)$$

z-Transform – Complex Plane



The complex variable z is defined as,

$$z = u + jv = r\,e^{j\omega}$$

where, u = Real part of z ; v = Imaginary part of z

$$r = \sqrt{u^2 + v^2} = \text{Magnitude of } z$$

$$\omega = \tan^{-1}\frac{v}{u} = \text{Phase or Argument of } z$$

jv
z-plane
z_1
r_1
−u
u
−jv

Fig 7.1 : z-plane.

The u and v take values from −∞ to +∞. A two dimensional complex plane with values of u on horizontal axis and values of v on vertical axis as shown in fig 7.1 is called z-plane. A circle with radius r_1 in z-plane represents all values of z_1 having same magnitude r_1 with variable phase (i.e., $\omega = 0$ to 2π).

Bilateral vs. Unilateral

- Two sided or bilateral z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Unilateral z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

  - also known as right sided z-transform

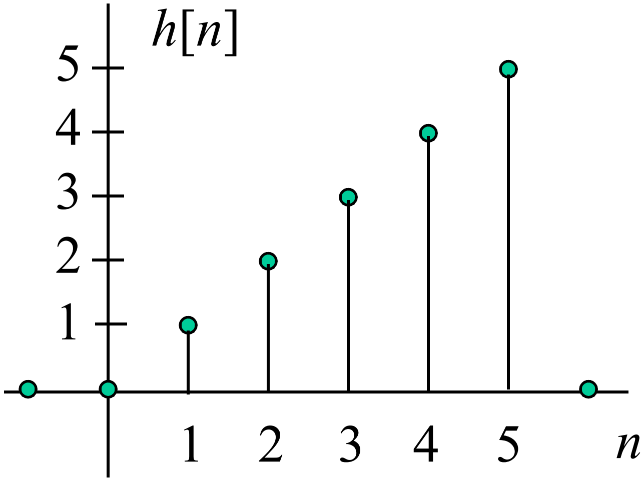
Applying Z-transform Definition

- **Finite extent signal**

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \sum_{n=0}^{5} h[n] z^{-n}$$

Polynomial in z^{-1}

$$H(z) = 0 + z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 5z^{-5}$$



- **Five-tap delay line**

Impulse response $h[n]$

Transfer function $H(z)$

$h[n] = n\ (\ u[n] - u[n\text{-}6]\ )$

Find z-transform

n	$n\leq -1$	0	1	2	3	4	5	$N>5$
$x[n]$	0	2	4	6	4	2	1	0

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = X(z)$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

The z-Transform

- Counterpart of the Laplace transform for discrete-time signals
- Generalization of the Fourier Transform
  - Fourier Transform does not exist for all signals
- The z-Transform is often time more convenient to use
- Definition:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Compare to DTFT definition:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z is a complex variable that can be represented as $z=r\,e^{j\omega}$
- Substituting $z=e^{j\omega}$ will reduce the z-transform to DTFT

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GEOMETRIC SERIES

Name	**Sum**	**Condition**
Finite on [N_1, N_2]	$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$	none
Finite on [0, $N - 1$]	$\sum_{k=0}^{N} a^k = \frac{1-a^{N+1}}{1-a}$	none
Infinite	$\sum_{n=0}^{\infty} a^k = \frac{1}{1-a}$	$\lvert a \rvert < 1$

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Convergence of the z-Transform

- DTFT does not always converge

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

  - Infinite sum not always finite if x[n] no absolute summable
  - **Example**: $x[n] = a^n u[n]$ for $|a|>1$ does not have a DTFT

- Complex variable z can be written as $r\,e^{j\omega}$ so the z-transform

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]\left(\mathrm{r}e^{-j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]\mathrm{r}^{-n}\right)e^{-j\omega n}$$

- DTFT of x[n] multiplied with exponential sequence r^{-n}
  - For certain choices of r the sum maybe made finite

$$\sum_{n=-\infty}^{\infty} \left|x[n]\, r^{-n}\right| < \infty$$

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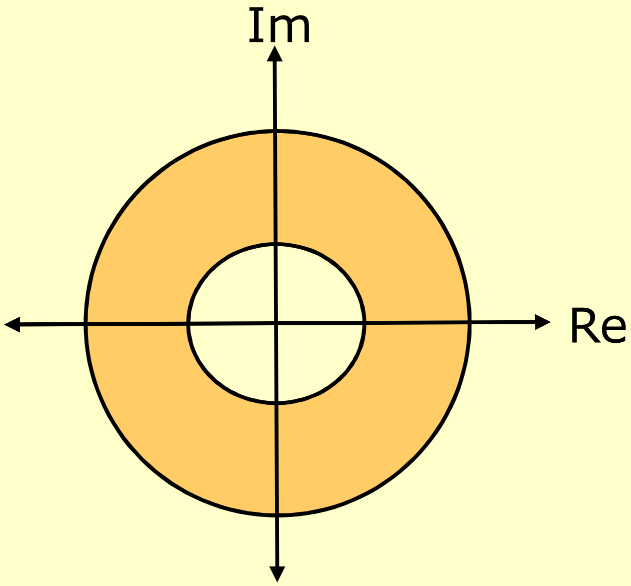
Region of Convergence

- The set of values of z for which the z-transform converges
- Each value of r represents a circle of radius r
- The region of convergence is made of circles



- Example: z-transform converges for values of 0.5<r<2
  - ROC is shown on the left
  - In this example the ROC includes the unit circle, so DTFT exists
- Not all sequence have a z-transform
- Example: $x[n] = \cos(\omega_o n)$
  - Does not converge for any r
  - No ROC, No z-transform
  - But DTFT exists?!
  - Sequence has finite energy
  - DTFT converges in the mean-squared sense

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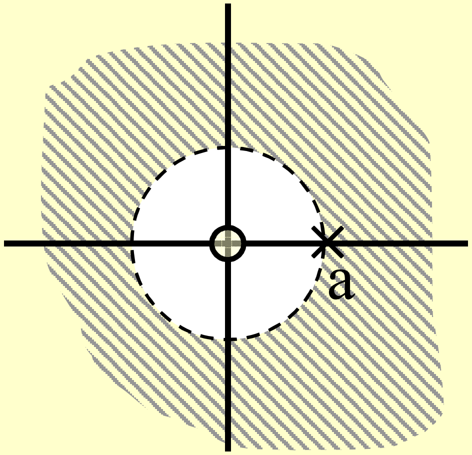
Right-Sided Exponential Sequence Example

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

- For Convergence we require

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty$$



- From infinite geometric series formula, we get,

$$\left|az^{-1}\right|^n < 1 \Rightarrow |z| > |a|$$

- Inside the ROC series converges to

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

- Region outside the circle of radius a is the ROC
- Right-sided sequence ROCs extend outside a circle

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Example: Sum of Two Exponential Sequences

Given $$x(n)=\left(\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{3}\right)^n u(n)$$

Then $$X(z)=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u(n)z^{-n}+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}\right)^n u(n)z^{-n}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}+\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^{-n}$$

$$=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}}=\frac{2z\left(z-\frac{1}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

Example: Sum of Two Exponential Sequences (continue)

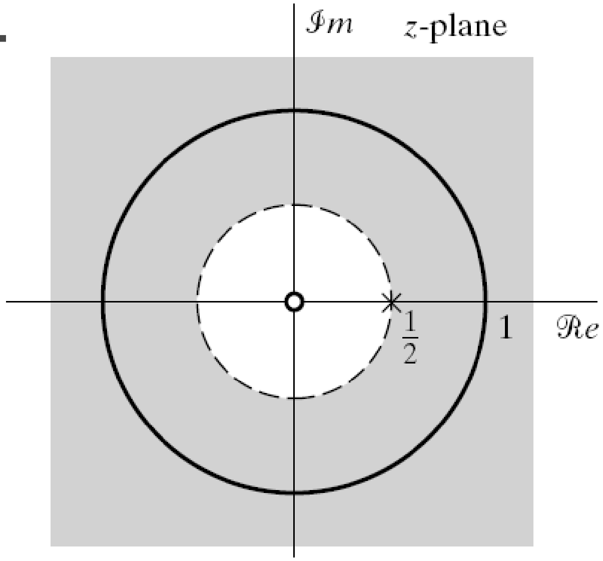
$$\left(\frac{1}{2}\right)^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

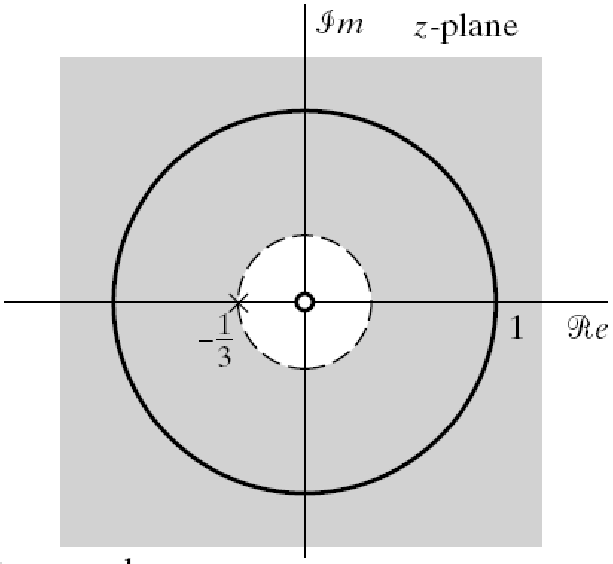
$$\left(-\frac{1}{3}\right)^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

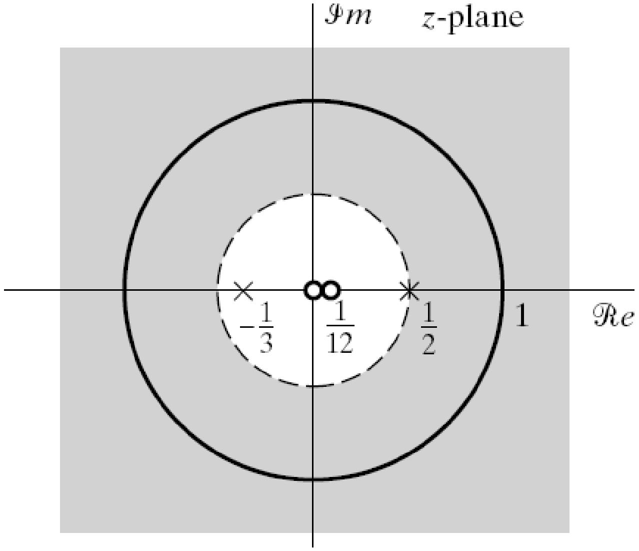
Thus

$$\left(\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{3}\right)^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{2}$$

Example: Sum of Two Exponential Sequences (continue)







Example: Left-sided Exponential Sequence

- Left-sided sequence:
  - A discrete-time signal is left-sided if it is nonzero only for $n \leq -1$.
- Consider the signal $x[n] = -a^n u[-n-1]$.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

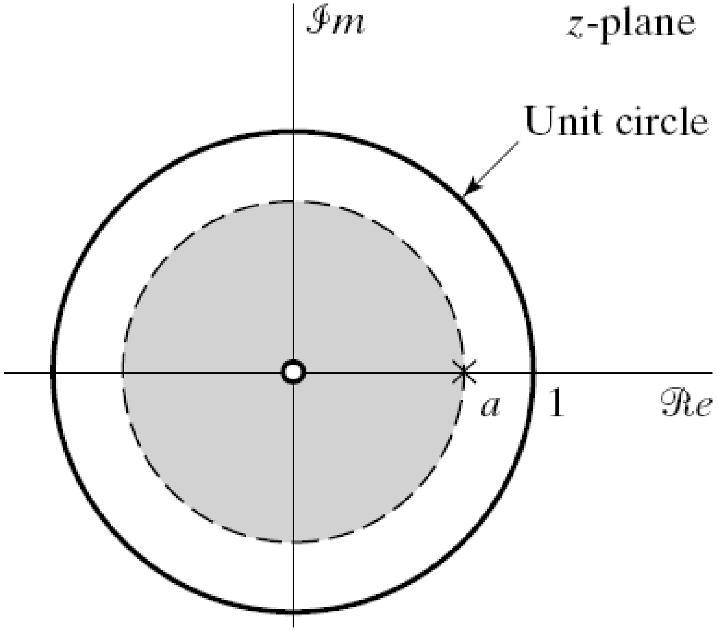
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n$$

  - If $|a^{-1}z| < 1$ or, equivalently, $|z| < a$, the sum converges.

Example: Left-sided Exponential Sequence (continue)

- By sum of power series,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}, \quad |z| < |a|$$



The pole-zero plot and the algebraic expression of the system function are the same as those in the previous example, but the ROC is different.

Two-Sided Exponential Sequence Example

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$$

$$\sum_{n=0}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n = \frac{\left(-\frac{1}{3}z^{-1}\right)^0 - \left(-\frac{1}{3}z^{-1}\right)^\infty}{1+\frac{1}{3}z^{-1}} = \frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\text{ROC}: \left|-\frac{1}{3}z^{-1}\right| < 1$$

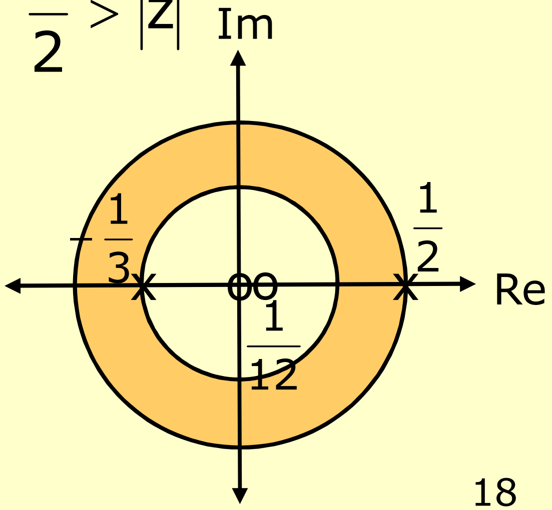
$$\frac{1}{3} < |z|$$

$$\sum_{n=-\infty}^{-1}\left(\frac{1}{2}z^{-1}\right)^n = \frac{\left(\frac{1}{2}z^{-1}\right)^{-\infty} - \left(\frac{1}{2}z^{-1}\right)^0}{1-\frac{1}{2}z^{-1}} = \frac{-1}{1-\frac{1}{2}z^{-1}}$$

$$\text{ROC}: \left|\frac{1}{2}z^{-1}\right| > 1$$

$$\frac{1}{2} > |z|$$

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$



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Example: Finite-length Sequence (FIR System)

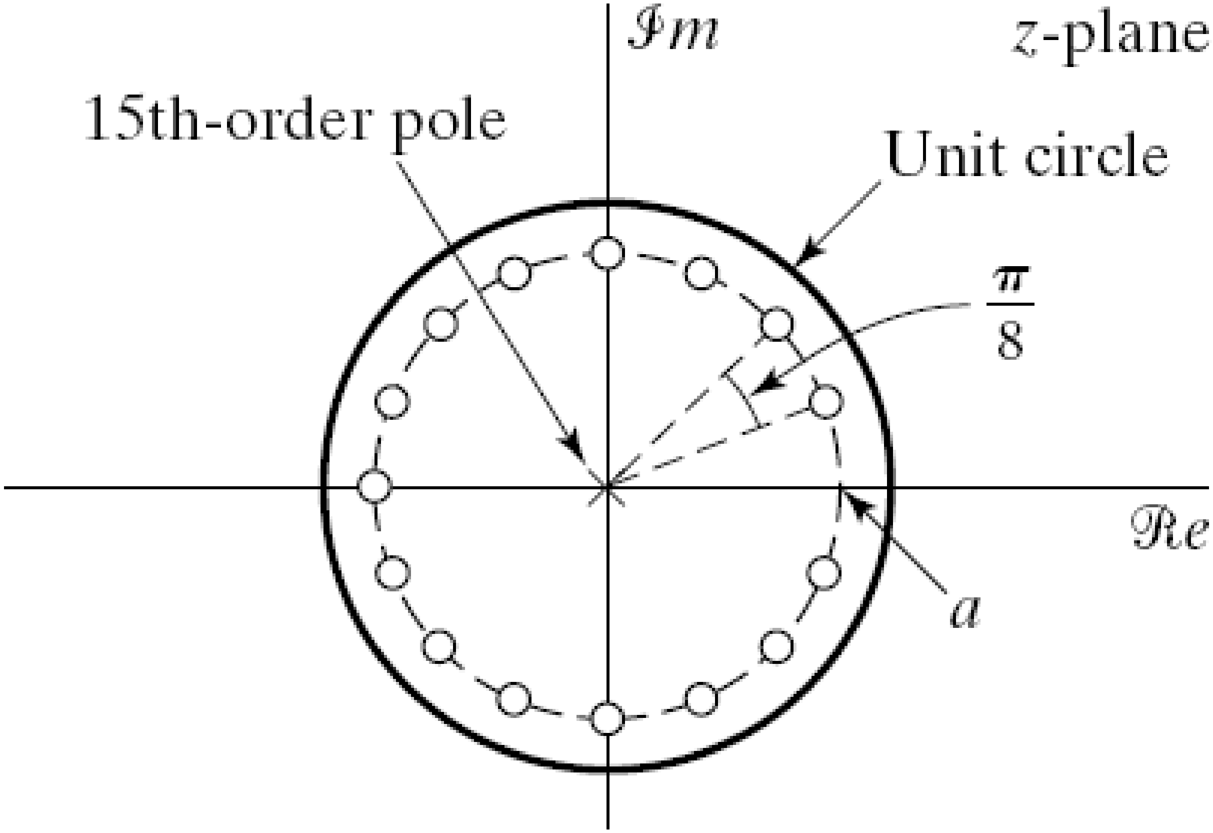
Given $$x(n)=\begin{cases} a^n & 0\le n\le N-1 \\ 0 & otherwise \end{cases}$$

Then $$X(z)=\sum_{n=0}^{N-1}a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}$$

$$=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}$$

There are the N roots of $z^N = a^N$, $z_k = ae^{j(2\pi k/N)}$. The root of $k = 0$ cancels the pole at $z=a$. Thus there are N-1 zeros, $z_k = ae^{j(2\pi k/N)}$, $k = 1 \ldots N$, and a (N-1)th order pole at origin.

Pole-zero Plot



Z-Transform Pairs

Sequence	z-Transform	ROC
$\delta(n)$	1	All z
$\delta(n-m)$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$

Z-Transform Pairs

Sequence	z-Transform	ROC
$[\cos \omega_0 n]u(n)$	$\dfrac{1-[\cos \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z \rvert > 1$
$[\sin \omega_0 n]u(n)$	$\dfrac{[\sin \omega_0]z^{-1}}{1-[2\cos \omega_0]z^{-1}+z^{-2}}$	$\lvert z \rvert > 1$
$[r^n \cos \omega_0 n]u(n)$	$\dfrac{1-[r\cos \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z \rvert > r$
$[r^n \sin \omega_0 n]u(n)$	$\dfrac{[r\sin \omega_0]z^{-1}}{1-[2r\cos \omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z \rvert > r$
$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z \rvert > 0$

Determine z Transform & ROC

$$x[n] = \{\underset{\uparrow}{3}\ \ 2\ \ 5\ \ 7\}$$

$$X(z) = \sum_{n=0}^{3} x(n)\, z^{-n}$$

$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3}$$

$$= 3 + 2z^{-1} + 5z^{-2} + 7z^{-3}$$

$$= 3 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3}$$

In X(z), when z=0, expect the first term all the other terms will become infinite. Hence X(z) will be finite for all values of z, expect z = 0. Therefore the **ROC is the entire z-plane expect z = 0**.

Determine z Transform & ROC

$$x[n] = \{6 \quad 4 \quad 5 \quad \underset{\uparrow}{3}\}$$

$$X(z) = \sum_{n=-3}^{0} x(n)\, z^{-n}$$

$$= x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z + x(0)$$

$$= 6z^{3} + 4z^{2} + 5z + 3$$

In X(z), when z=∞, expect the last term all the other terms will become infinite. Hence X(z) will be finite for all values of z, expect z = ∞. Therefore the **ROC is the entire z-plane expect z = ∞**.

Determine z Transform & ROC

$$x[n] = \{2 \;\; 4 \;\; \underset{\uparrow}{5} \;\; 7 \;\; 3\}$$

$$X(z) = \sum_{n=-2}^{2} x(n)\, z^{-n}$$

$$= x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2}$$

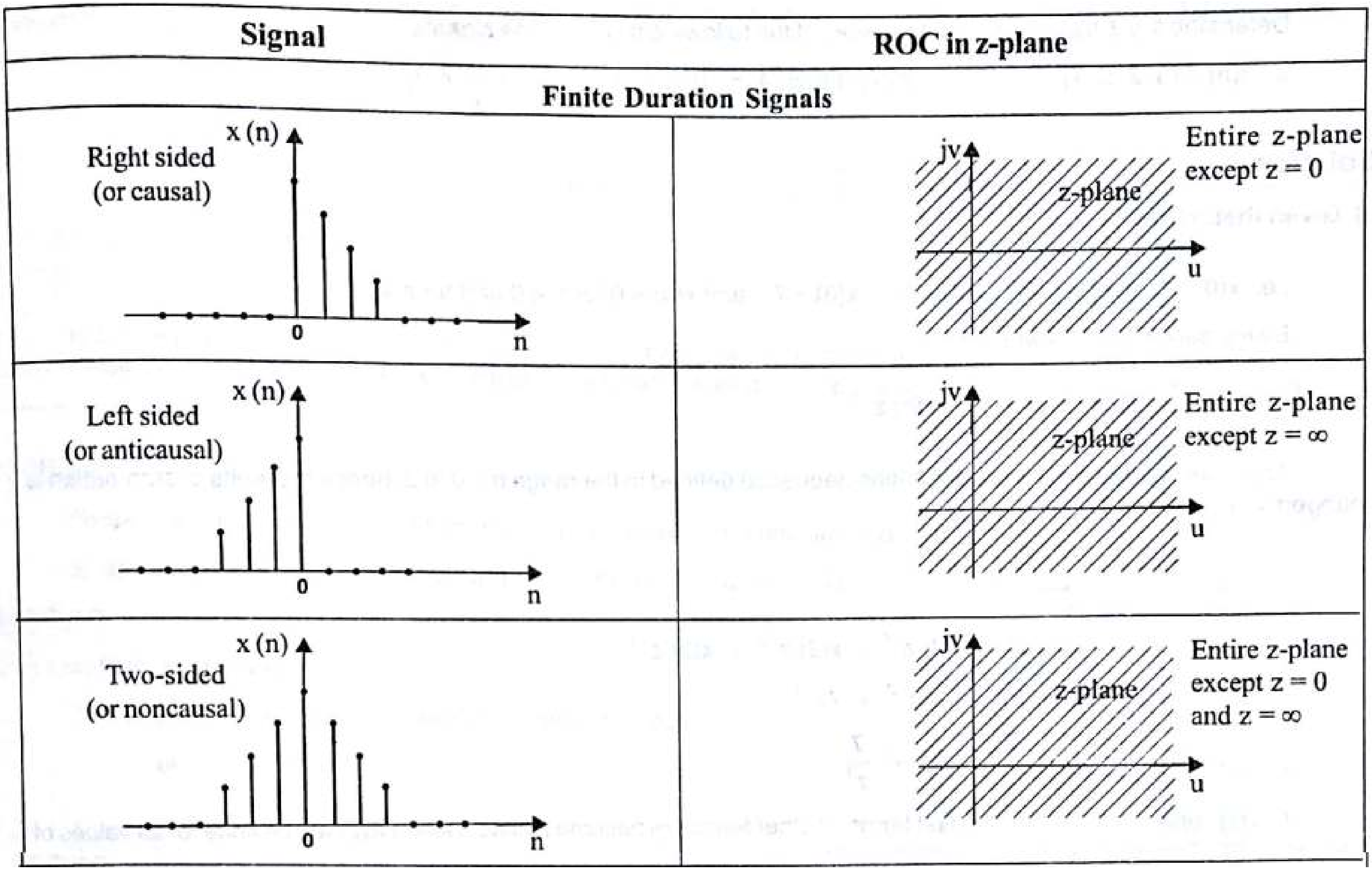
$$= 2z^{2} + 4z + 5 + 7z^{-1} + 3z^{-2}$$

$$= 2z^{2} + 4z + 5 + \frac{7}{z} + \frac{3}{z^{2}}$$

ROC is the entire z-plane expect

z = 0 and z = ∞.

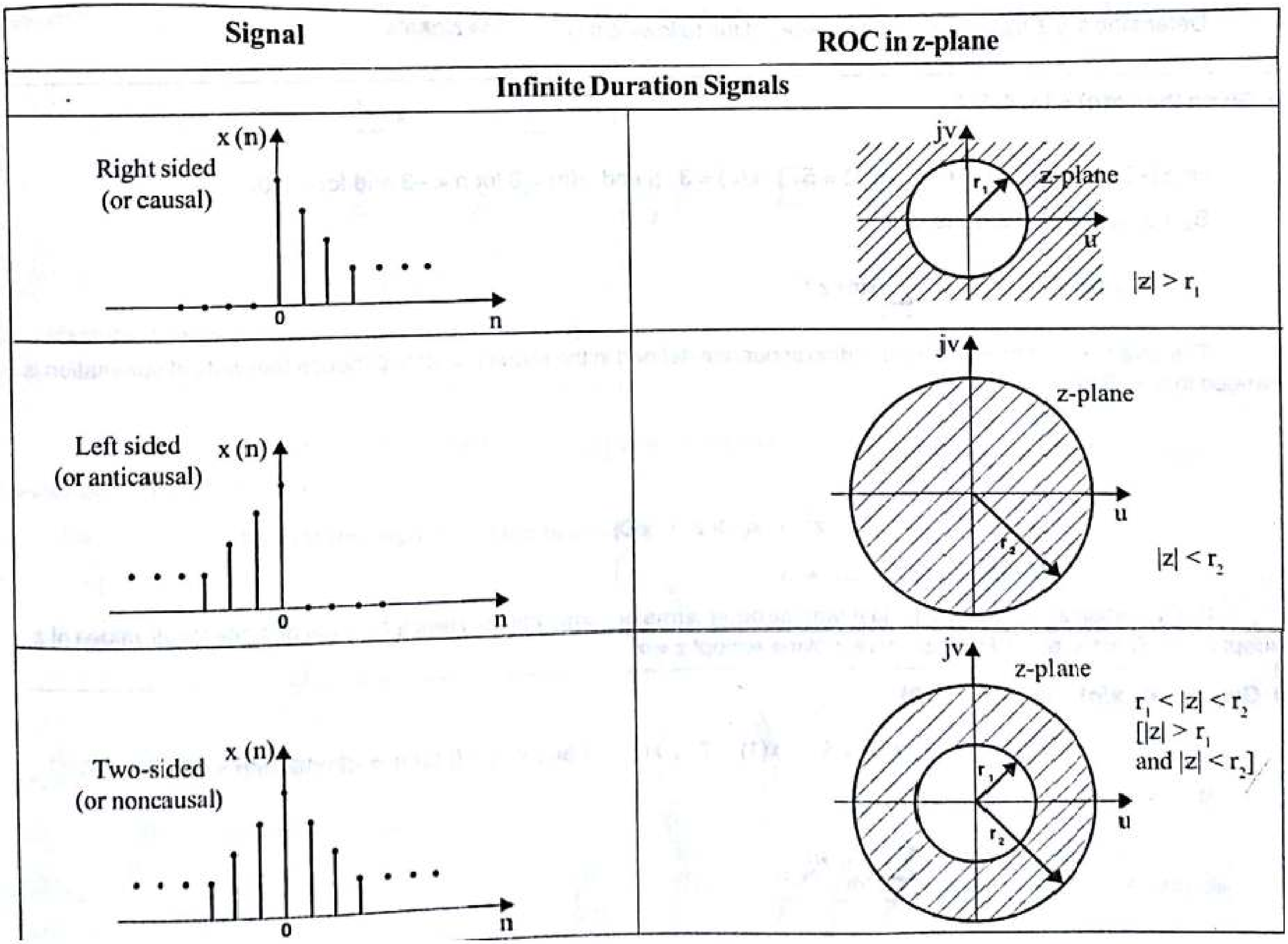
Signal characteristics and their ROC



Signal	ROC in z-plane
Finite Duration Signals	
Right sided (or causal)	Entire z-plane except z = 0
Left sided (or anticausal)	Entire z-plane except z = ∞
Two-sided (or noncausal)	Entire z-plane except z = 0 and z = ∞

x (n)
0
n
jv
z-plane
u

Signal characteristics and their ROC



Signal	ROC in z-plane
Infinite Duration Signals	
Right sided (or causal)	$\lvert z\rvert > r_1$
Left sided (or anticausal)	$\lvert z\rvert < r_2$
Two-sided (or noncausal)	$r_1 < \lvert z\rvert < r_2$ $[\lvert z\rvert > r_1$ and $\lvert z\rvert < r_2]$

x (n)
0
n
jv
z-plane
u
r_1
r_2

Properties of the ROC

- The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \le r_R < |z| < r_L \le \infty$.
- The Fourier transform of $x[n]$ converges absolutely iff the ROC includes the unit circle.
- The ROC cannot contain any poles
- If $x[n]$ is a finite duration sequence, then the ROC is the entire z-plane except possible $z = 0$ or $z = \infty$.
- If $x[n]$ is a right-sided sequence, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly include) $z = \infty$.

Properties of the ROC (continue)

- If $x[n]$ is a left-sided sequence, the ROC extends inward from the innermost (i.e., smallest magnitude) nonzero pole in $X(z)$ to (and possibly include) $z = 0$.
- A two-sided sequence $x[n]$ is an infinite-duration sequence that is neither right nor left sided. The ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole, but not containing any poles.
- The ROC must be a connected region.

BIBO Stability

- **Rule #1: Poles inside unit circle (causal signals)**
- **Rule #2: Unit circle in region of convergence**

  Analogy in continuous-time: imaginary axis would be in region of convergence of Laplace transform

- **Example:** $a^n u[n] \overset{z}{\leftrightarrow} \dfrac{1}{1 - a\, z^{-1}}$ for $|z| > |a|$

  BIBO stable if $|a| < 1$ by rule #1

  BIBO stable if $|z| > |a|$ includes unit circle; hence, $|a| < 1$ by rule #2

BIBO means Bounded-Input Bounded-Output

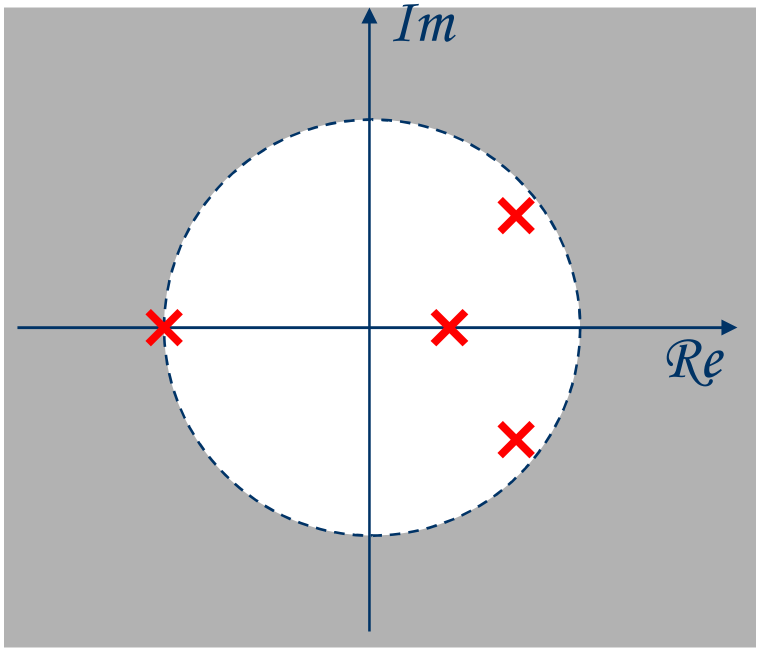
Properties of the ROC (continue)

- Consider the system function $H(z)$ of a linear system:
  - If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle,.
  - If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z = \infty$.

Stable and Causal Systems

Causal Systems : ROC extends outward from the outermost pole.

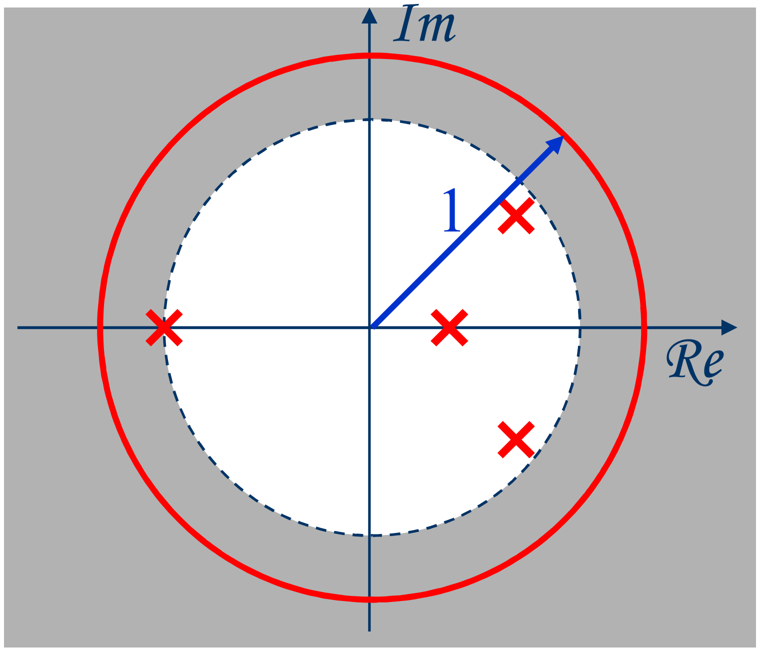
$$H(z)=\frac{A\prod_{r=1}^{M}(1-c_r z^{-1})}{\prod_{k=1}^{N}(1-d_r z^{-1})}$$



Stable and Causal Systems

Stable Systems : ROC includes the unit circle.

$$H(z) = \frac{A\prod_{r=1}^{M}(1 - c_r z^{-1})}{\prod_{k=1}^{N}(1 - d_r z^{-1})}$$



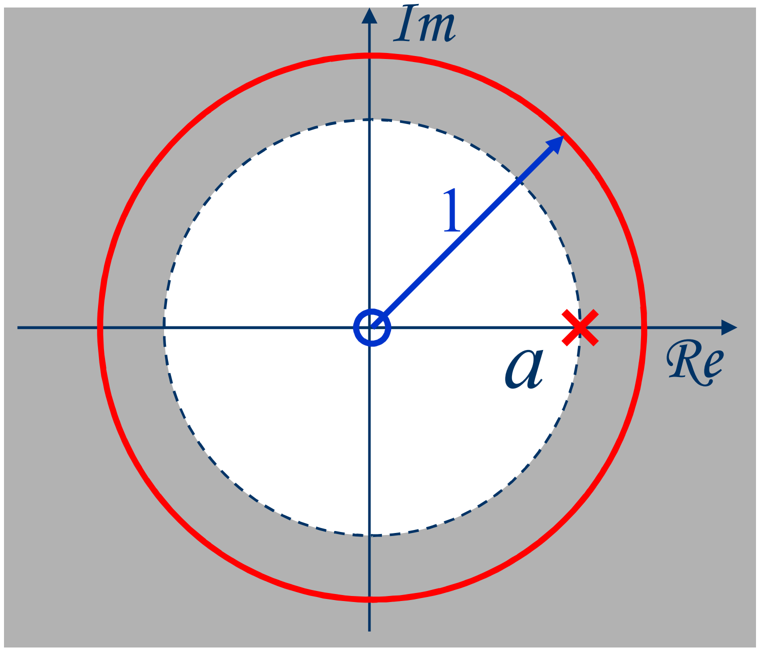
Example

Consider the causal system characterized by

$$y(n) = ay(n-1) + x(n)$$

$$\boxed{H(z) = \frac{1}{1-az^{-1}}}$$

$$h(n) = a^n u(n)$$



Inverse z-Transform

The inverse z-transform can be derived by using Cauchy's integral theorem. Start with the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Multiply both sides by $z^{k\text{-}1}$ and integrate with a contour integral for which the contour of integration encloses the origin and lies entirely within the region of convergence of $X(z)$:

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = \frac{1}{2\pi i}\oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+k-1} dz$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi i}\oint_C z^{-n+k-1} dz$$

$$\frac{1}{2\pi i}\oint_C X(z) z^{k-1} dz = x(n) \text{ is the inverse z - transform.}$$

Inverse Z-transform

- Techniques for finding the inverse z-transform:
  - Investigation method:
    - By inspect certain transform pairs.
    - Eg. If we need to find the inverse z-transform of

$$X(z) = \frac{1}{1 - 0.5z^{-1}}$$

From the transform pair we see that $x[n] = 0.5^n u[n]$.

- We need to specify both algebraic expression and ROC to make the inverse Z-transform unique.

ROC: |z| > |0.5|

Example

- Find the inverse z-transform of

$$X(z)=\frac{1}{\left(1-(1/4)z^{-1}\right)\left(1-(1/2)z^{-1}\right)} \qquad |z|>\frac{1}{2}$$

$X(z)$ can be decomposed as

$$X(z)=\frac{A_1}{\left(1-(1/4)z^{-1}\right)}+\frac{A_2}{\left(1-(1/2)z^{-1}\right)}$$

Then

$$A_1=\left(1-(1/4)z^{-1}\right)X(z)\Big|_{z=1/4}=-1$$

$$A_2=\left(1-(1/2)z^{-1}\right)X(z)\Big|_{z=1/2}=2$$

Example (continue)

- Thus

$$X(z) = \frac{-1}{\left(1-(1/4)z^{-1}\right)} + \frac{2}{\left(1-(1/2)z^{-1}\right)}$$

and so

$$x[n] = 2\left(\frac{1}{2}\right)^{n} u[n] - \left(\frac{1}{4}\right)^{n} u[n]$$

Inverse *z*-transform

- **Definition** $h[n] = \frac{1}{2\pi j} \oint_R H(z)\, z^{-n+1} dz$
- **Using the definition requires a contour integration in the complex *z*-plane**

  Use Cauchy residue theorem (from complex analysis) OR

  Use transform tables and transform pairs?
- **Fortunately, we tend to be interested in only a few basic signals (pulse, step, etc.)**

  Virtually all signals can be built from these basic signals

  For common signals, *z*-transform pairs have been tabulated

Example

$$X(z)=\frac{z^2+2z+1}{z^2-\frac{3}{2}z+\frac{1}{2}}$$

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$X(z)=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

- **Ratio of polynomial z-domain functions**
- **Divide through by the highest power of z**
- **Factor denominator into first-order factors**
- **Use partial fraction decomposition to get first-order terms**

Example (con’t)

- **Find B_0 by polynomial division**

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1\big)\overline{z^{-2}+2z^{-1}+1} \\ \underline{z^{-2}-3z^{-1}+2} \\ 5z^{-1}-1 \end{array}$$

- **Express in terms of B_0**

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

- **Solve for A_1 and A_2**

$$A_1=\left.\frac{1+2z^{-1}+z^{-2}}{1-z^{-1}}\right|_{z^{-1}=2}=\frac{1+4+4}{1-2}=-9$$

$$A_2=\left.\frac{1+2z^{-1}+z^{-2}}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=1}=\frac{1+2+1}{\frac{1}{2}}=8$$

Example (con't)

- **Express** $X(z)$ **in terms of** B_0**,** A_1**, and** A_2

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2} z^{-1}} + \frac{8}{1 - z^{-1}}$$

- **Use table to obtain inverse** ***z*****-transform**

$$x[n] = 2\,\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8\,u[n]$$

- **With the unilateral** ***z*****-transform, or the bilateral** ***z*****-transform with region of convergence, the inverse** ***z*****-transform is unique**

Inverse *z*-transform – Residue Method

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$= \frac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)}$$

$$= \frac{3z^2 + 2z + 1}{z^2 - 3z + 2}$$

$$= 3 + \frac{11z - 5}{z^2 - 3z + 2}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline z^2 - 3z + 2 & 3z^2 + 2z + 1 \\ & 3z^2 - 9z + 6 \\ & (-) \quad (+) \quad (-) \\ \cline{2-2} & \qquad 11z - 5 \\ \cline{2-2} \end{array}$$

$$x(n) = \mathcal{Z}^{-1}\{3\} + \mathcal{Z}^{-1}\{X_2(z)\}$$

$$= 3\,\delta(n) + \sum_{i=1}^{N} \left[(z - p_i)\, X_2(z)\, z^{n-1} \Big|_{z = p_i} \right]$$

Inverse *z*-transform – Residue Method

$$x(n) = 3\,\delta(n) + \sum_{i=1}^{N}\left[(z - p_i)\,X_2(z)\,z^{n-1}\Big|_{z=p_i}\right]$$

Using residue theorem

$$= 3\,\delta(n) + (z-1)\,\frac{11z-5}{(z-1)(z-2)}\,z^{n-1}\Big|_{z=1}$$

$$+ (z-2)\,\frac{11z-5}{(z-1)(z-2)}\,z^{n-1}\Big|_{z=2}$$

$$= 3\,\delta(n) + \frac{11-5}{1-2}\,(1)^{n-1} + \frac{11\times 2-5}{2-1}\,2^{n-1}$$

$$= 3\,\delta(n) - 6\,u(n-1) + 17(2)^{n-1}\,u(n-1)$$

$$= 3\,\delta(n) + \left[-6 + 17(2)^{n-1}\right]u(n-1)$$

When $n = 0, 1, 2,$

$$x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots\ldots\}$$

Inverse *z*-transform – PFE

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$= \frac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \frac{3z^2 + 2z + 1}{(z - 1)(z - 2)}$$

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z - 1)(z - 2)} = \frac{A_1}{z} + \frac{A_2}{z - 1} + \frac{A_3}{z - 2}$$

$$A_1 = z\frac{X(z)}{z}\bigg|_{z=0} = \frac{1}{(-1)\times(-2)} = 0.5$$

$$A_2 = (z - 1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{3 + 2 + 1}{1 \times (1 - 2)} = -6$$

$$A_3 = (z - 2)\frac{X(z)}{z}\bigg|_{z=2} = \frac{3 \times 2^2 + 2 \times 2 + 1}{2 \times (2 - 1)} = 8.5$$

Inverse *z*-transform – PFE

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$X(z) = 0.5 - 6\,\frac{z}{z-1} + 8.5\,\frac{z}{z-2}$$

taking inverse z-transform of X(z) we get,

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5\,(2)^{n}u(n)$$

$$= 0.5\,\delta(n) + [-6 + 8.5(2)^{n}]\,u(n)$$

When n = 0, 1, 2,

$$x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots\}$$

$$Z\{\delta(n)\} = 1$$

$$Z\{u(n)\} = \frac{z}{z-1}$$

$$Z\{a^{n}u(n)\} = \frac{z}{z-a}$$

Inverse z-transform – Series Expansion

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$
\begin{array}{r|l}
 & 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots\ldots \\
\cline{2-2}
1 - 3z^{-1} + 2z^{-2} & 3 + 2z^{-1} + z^{-2} \\
 & 3 - 9z^{-1} + 6z^{-2} \\
 & (-)\quad (+)\quad (-) \\
\cline{2-2}
 & 11z^{-1} - 5z^{-2} \\
 & 11z^{-1} - 33z^{-2} + 22z^{-3} \\
 & (-)\quad (+)\quad (-) \\
\cline{2-2}
 & 28z^{-2} - 22z^{-3} \\
 & 28z^{-2} - 84z^{-3} + 56z^{-4} \\
 & (-)\quad (+)\quad (-) \\
\cline{2-2}
 & 62z^{-3} - 56z^{-4} \\
 & 62z^{-3} - 186z^{-4} + 124z^{-5} \\
 & (-)\quad (+)\quad (-) \\
\cline{2-2}
 & 130z^{-4} - 124z^{-5} \\
 & \vdots
\end{array}
$$

Inverse *z*-transform – Series Expansion

$$X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$$

$$= 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots..$$

by definition of z-transform,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$= \ldots..+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + \ldots...$$

On comparing equations (1) and (2) we get,

$$x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots..\}$$

Inverse *z*-transform –Example

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{2}{z+1} + \frac{4.5}{z+2}$$

$$X(z) = 0.5 - \frac{2z}{z+1} + \frac{4.5z}{z+2}$$

$$= 0.5 - 2\frac{z}{z-(-1)} + 4.5\frac{z}{z-(-2)}$$

$$x(n) = 0.5\,\delta(n) - 2(-1)^n u(n) + 4.5(-2)^n u(n)$$

Inverse *z*-transform –Example

$$X(z) = \frac{z - 4}{(z - 1)(z - 2)^2}$$

$$= \frac{A_1}{z - 1} + \frac{A_2}{(z - 2)^2} + \frac{A_3}{(z - 2)}$$

$$= \frac{-3}{z - 1} - \frac{2}{(z - 2)^2} + \frac{3}{z - 2}$$

$$= -3z^{-1}\frac{z}{z - 1} - z^{-1}\frac{2z}{(z - 2)^2} + 3z^{-1}\frac{z}{z - 2}$$

$$x(n) = -3\,u(n-1) - (n-1)\,2^{n-1}\,u(n-1) + 3 \times 2^{n-1}\,u(n-1)$$

Inverse *z*-transform –Example

$$\mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z-a}$$

$$\mathcal{Z}\{na^n u(n)\} = \frac{az}{(z-a)^2}$$

$$\mathcal{Z}\{x(n-1)\} = z^{-1} X(z)$$

$$\mathcal{Z}\{a^{(n-1)} u(n-1)\} = z^{-1}\frac{z}{z-a}$$

$$\mathcal{Z}\{(n-1)\, a^{(n-1)} u(n-1)\} = z^{-1}\frac{az}{(z-a)^2}$$